

# HOM-LIE TRIPLE SYSTEM AND HOM-BOL ALGEBRA STRUCTURES ON HOM-MALCEV AND RIGHT HOM-ALTERNATIVE ALGEBRAS

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## Abstract

Every multiplicative Hom-Malcev algebra has a natural multiplicative Hom-Lie triple system structure. Moreover, there is a natural Hom-Bol algebra structure on every multiplicative Hom-Malcev algebra and on every multiplicative right (or left) Hom-alternative algebra.

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## 1. Introduction

The study of Lie triple systems (Lts) on their own as algebraic objects started from Jacobson's work [11] and developed further by, e.g., Lister [13], Yamaguti [29] and other mathematicians. The interplay between Lts and the differential geometry of symmetric spaces is now folk (see, e.g., [12], [15]). Lts constitute examples of ternary algebras. If  $(g, [,])$  is a Lie algebra, then  $(g, [, ,])$  is a Lts, where  $[x, y, z] := [[x, y], z]$  (see [11], [12], [15]). Another construction of Lts from binary algebras is the one from Malcev algebras found by Loos [14].

Malcev algebras were introduced by Mal'tsev [20] in a study of commutator algebras of alternative algebras and also as a study of tangent algebras to local smooth Moufang loops. Mal'tsev used the name "Moufang-Lie algebras" for these nonassociative algebras while Sagale [25] introduced the term "Malcev algebras". Equivalent defining identities of Malcev algebras are pointed out in [25].

Alternative algebras, Malcev algebras and Lts (among other algebras) received a twisted generalization in the development of the theory of Hom-algebras during these latest years. The forerunner of the theory of Hom-algebras is the Hom-Lie algebra introduced by Hartwig, Larsson and Silvestrov in [7] in order to describe the structure of some deformation of the Witt algebra and the Virasoro algebra. It is well-known that Lie algebras are related to associative algebras via the commutator bracket construction. In the search of a similar construction for Hom-Lie algebras, the notion of a Hom-associative algebra is introduced by Makhlouf and Silvestrov in [18], where it is proved that a Hom-associative algebra gives rise to a Hom-Lie algebra via the commutator bracket construction. Since then, various Hom-type structures are considered (see, e.g., [1], [2], [4]-[6], [9], [16]-[19], [32]-[35]). Roughly speaking, Hom-algebraic structures are corresponding ordinary algebraic structures whose defining identities are twisted by a linear self-map. A general method for constructing a Hom-type algebra from the ordinary type of algebra with a linear self-map is given by Yau in [31].

In [1], [33],  $n$ -ary Hom-algebra structures generalizing  $n$ -ary algebras of Lie type or associative type were considered. In particular, generalizations of  $n$ -ary Nambu or Nambu-Lie algebras, called  $n$ -ary Hom-Nambu and Hom-Nambu-Lie algebras respectively, were introduced in [1] while Hom-Jordan were defined in [17] and Hom-Lie triple systems (Hom-Lts) were introduced in [33] (here, another definition of a Hom-Jordan algebra is given). It is shown [33] that Hom-Lts are ternary Hom-Nambu algebras with additional properties, that Hom-Lts arise also from Hom-Jordan triple systems or from other Hom-type algebras.

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Motivated by the relationships between some classes of binary algebras and some classes of binary-ternary algebras, a study of Hom-type generalization of binary-ternary algebras is initiated in [9] with the definition of Hom-Akivis algebras. Further, Hom-Lie-Yamaguti algebras are considered in [5] and Hom-Bol algebras [2] are defined as a twisted generalization of Bol algebras which are introduced and studied in [21], [26], [27] as infinitesimal structures tangent to smooth Bol loops (some aspects of the theory of Bol algebras are discussed in [3], [8] and [24]).

In this paper, we will be concerned with right (or left) Hom-alternative algebras, Hom-Malcev algebras and Hom-Bol algebras. We extend the Loos' construction of Lts from Malcev algebras ([14], Satz 1) to the Hom-algebra setting (Section 3). Specifically, we prove (Theorem 3.2) that every multiplicative Hom-Malcev algebra is naturally a multiplicative Hom-Lts by a suitable definition of the ternary operation. As a tool in the proof of this fact, we point out a kind of compatibility relation between the original binary operation of a given Hom-Malcev algebra and the ternary operation mentioned above (Lemma 3.1). Moreover, we obtain that every multiplicative Hom-Malcev algebra has a natural Hom-Bol algebra structure (Theorem 3.5). In [22] Mikheev proved that every right alternative algebra has a natural (left) Bol algebra structure. In [8] Hentzel and Peresi proved that not only a right alternative algebra but also a left alternative algebra have left Bol algebra structure. In Section 4 we prove that the Hom-analogue of these results hold. Specifically, every multiplicative right (or left) Hom-alternative algebra is a Hom-Bol algebra (Theorem 4.4). In Section 2 we recall some basic definitions and facts about Hom-algebras. We define the Hom-Jordan associator of a given Hom-algebra and point out that every Hom-algebra is a Hom-triple system with respect to the Hom-Jordan associator. This observation is used in the proof of Theorem 4.4.

All vector spaces and algebras are meant over an algebraically closed ground field  $\mathbb{K}$  of characteristic 0.

## 2. Some basics on Hom-algebras

We first recall some relevant definitions about binary and ternary Hom-algebras. In particular, we recall the notion of a Hom-Malcev algebra as well as some of its equivalent defining identities. Although various types of  $n$ -ary Hom-algebras are introduced and discussed in [1], [33], for our purpose, we will consider ternary Hom-algebras (ternary Hom-Nambu algebras and Hom-Lts) and Hom-Bol algebras. For fundamentals on Hom-algebras, one may refer, e.g., to [1], [4], [7], [16], [18], [30], [31]. Some aspects of the theory of binary Hom-algebras are considered in [28], while some classes of binary-ternary Hom-algebras are defined and discussed in [2], [5], [9].

**Definition 2.1.** (i) A *Hom-algebra* is a triple  $(A, *, \alpha)$  in which  $A$  is a  $\mathbb{K}$ -vector space,  $*$  :  $A \times A \longrightarrow A$  a bilinear map (the binary operation) and  $\alpha : A \longrightarrow A$  a linear map (the twisting map). The Hom-algebra  $A$  is said to be *multiplicative* if  $\alpha(x * y) = \alpha(x) * \alpha(y)$  for all  $x, y \in A$ .

(ii) The *Hom-Jacobian* in  $(A, *, \alpha)$  is the trilinear map  $J_\alpha : A \times A \times A \longrightarrow A$  defined as  $J_\alpha(x, y, z) := \circ_{x,y,z} (x * y) * \alpha(z)$ , where  $\circ_{x,y,z}$  denotes the sum over cyclic permutation of  $x, y, z$ .

(iii) The *Hom-associator* of a Hom-algebra  $(A, *, \alpha)$  is the trilinear map  $as : A^{\otimes 3} \longrightarrow A$  defined as  $as(x, y, z) = (x * y) * \alpha(z) - \alpha(x) * (y * z)$ . If  $as(x, y, z) = 0$  for all  $x, y, z \in A$ , then  $(A, *, \alpha)$  is said to be *Hom-associative*.

*Remark 2.2.* If  $\alpha = id$  (the identity map), then a Hom-algebra  $(A, *, \alpha)$  reduces to an ordinary algebra  $(A, *)$ , the Hom-Jacobian  $J_\alpha$  is the ordinary Jacobian  $J$ , and the Hom-associator is the usual associator for the algebra  $(A, *)$ .

As for ordinary algebras, to each Hom-algebra  $\mathcal{A} := (A, *, \alpha)$  are attached two Hom-algebras: the *commutator Hom-algebra*  $\mathcal{A}^- := (A, [, \alpha)$ , where  $[x, y] := x * y - y * x$  (the commutator of  $x$  and  $y$ ), and the *plus Hom-algebra*  $\mathcal{A}^+ := (A, \circ, \alpha)$ , where  $x \circ y := x * y + y * x$

(the *Jordan product*) for all  $x, y \in A$ .

For our purpose, we make the following

**Definition 2.3.** The *Hom-Jordan associator* of a Hom-algebra  $\mathcal{A} := (A, *, \alpha)$  is the trilinear map  $as^J : A^{\otimes 3} \rightarrow A$  defined as  $as^J(x, y, z) = (x \circ y) \circ \alpha(z) - \alpha(x) \circ (y \circ z)$ , where " $\circ$ " is the Jordan product on  $A$ .

If  $\alpha = id$ , the Hom-Jordan associator reduces to the usual Jordan associator.

**Definition 2.4.** (i) A *Hom-Lie algebra* is a Hom-algebra  $(A, *, \alpha)$  such that the binary operation " $*$ " is anticommutative and the *Hom-Jacobi identity*

$$(2.1) \quad J_\alpha(x, y, z) = 0$$

holds for all  $x, y, z$  in  $A$  ([7]).

(ii) A *Hom-Malcev algebra* (or *Hom-Maltsev algebra*) is a Hom-algebra  $(A, *, \alpha)$  such that the binary operation " $*$ " is anticommutative and that the *Hom-Malcev identity*

$$(2.2) \quad J_\alpha(\alpha(x), \alpha(y), x * z) = J_\alpha(x, y, z) * \alpha^2(x)$$

holds for all  $x, y, z$  in  $A$  ([32]).

(iii) A *Hom-Jordan algebra* is a Hom-algebra  $(A, *, \alpha)$  such that  $(A, *)$  is a commutative algebra and the *Hom-Jordan identity*

$$as(x * x, \alpha(y), \alpha(x)) = 0$$

is satisfied for all  $x, y$  in  $A$  ([32]).

(iv) A Hom-algebra  $(A, *, \alpha)$  is called a *right Hom-alternative algebra* if

$$as(x, y, y) = 0$$

for all  $x, y$  in  $A$ . A Hom-algebra  $(A, *, \alpha)$  is called a *left Hom-alternative algebra* if

$$as(x, x, y) = 0$$

for all  $x, y$  in  $A$ . A Hom-algebra  $(A, *, \alpha)$  is called a *Hom-alternative algebra* if it is both right and left Hom-alternative ([17]).

*Remark 2.5.* When  $\alpha = id$ , the Hom-Jacobi identity (2.1) is the usual *Jacobi identity*  $J(x, y, z) = 0$ . Likewise, for  $\alpha = id$ , the Hom-Malcev identity (2.2) reduces to the *Malcev identity*  $J(x, y, x * z) = J(x, y, z) * x$ . Therefore a Lie (resp. Malcev) algebra  $(A, *)$  may be seen as a Hom-Lie (resp. Hom-Malcev) algebra with the identity map as the twisting map. Also Hom-Malcev algebras generalize Hom-Lie algebras in the same way as Malcev algebras generalize Lie algebras. For  $\alpha = id$  in the Hom-Jordan identity, we recover the usual Jordan identity. Observe that the definition of the Hom-Jordan identity in [32] is slightly different of the one formerly given in [17].

Hom-Malcev algebras are introduced in [32] in connection with a study of Hom-alternative algebras introduced in [17]. In fact it is proved ([32], Theorem 3.8) that every Hom-alternative algebra is *Hom-Malcev admissible*, i.e. the commutator Hom-algebra of any Hom-alternative algebra is a Hom-Malcev algebra (this is the Hom-analogue of Mal'tsev's construction of Malcev algebras as commutator algebras of alternative algebras [20]). This result is also mentioned in [9], section 4, using an approach via Hom-Akivis algebras (this approach is close to the one of Mal'tsev in [20]). Also, every Hom-alternative algebra is *Hom-Jordan admissible*, i.e. its plus Hom-algebra is a Hom-Jordan algebra ([32]). Examples of Hom-

alternative algebras and Hom-Jordan algebras could be found in [17] and [32]. An example of a right Hom-alternative algebra that is not left Hom-alternative is given in [35].

Equivalent to (2.2) defining identities of Hom-Malcev algebras are found in [32] where, in particular, it is shown that the identity

$$(2.3) \quad J_\alpha(\alpha(x), \alpha(y), w * z) + J_\alpha(\alpha(w), \alpha(y), x * z) = \\ J_\alpha(x, y, z) * \alpha^2(w) + J_\alpha(w, y, z) * \alpha^2(x)$$

is equivalent to (2.2) in any anticommutative Hom-algebra  $(A, *, \alpha)$  ([32], Proposition 2.7). In [10], it is proved that in any anticommutative Hom-algebra  $(A, *, \alpha)$ , the Hom-Malcev identity (2.2) is equivalent to

$$(2.4) \quad J_\alpha(\alpha(x), \alpha(y), u * v) = \alpha^2(u) * J_\alpha(x, y, v) + J_\alpha(x, y, u) * \alpha^2(v) \\ - 2J_\alpha(\alpha(u), \alpha(v), x * y).$$

Therefore, apart from (2.2), the identities (2.3) and (2.4) may be taken as defining identities of a Hom-Malcev algebra.

The Hom-algebras mentioned above are *binary* Hom-algebras. The first generalization of binary algebras was the ternary algebras introduced in [11]. Ternary algebraic structures also appeared in various domains of theoretical and mathematical physics (see, e.g., [23]). Likewise, binary Hom-algebras are generalized to  $n$ -ary Hom-algebra structures in [1] (see also [33]).

**Definition 2.6** ([1]). A *ternary Hom-Nambu algebra* is a triple  $(A, [, ], \alpha)$  in which  $A$  is a  $\mathbb{K}$ -vector space,  $[, , ] : A \times A \times A \longrightarrow A$  a trilinear map, and  $\alpha = (\alpha_1, \alpha_2)$  a pair of linear maps (the twisting maps) such that the identity

$$(2.5) \quad [\alpha_1(x), \alpha_2(y), [u, v, w]] = [[x, y, u], \alpha_1(v), \alpha_2(w)] + [\alpha_1(u), [x, y, v], \alpha_2(w)] \\ + [\alpha_1(u), \alpha_2(v), [x, y, w]]$$

holds for all  $u, v, w, x, y$  in  $A$ . The identity (2.5) is called the *ternary Hom-Nambu identity*.

*Remark 2.7.* When  $\alpha_1 = id = \alpha_2$  one recovers the usual ternary Nambu algebra. One may refer to [23] for the origins of Nambu algebras. In [1], examples of  $n$ -ary Hom-Nambu algebras that are not Nambu algebras are provided.

**Definition 2.8** ([33]). A *Hom-Lie triple system* (Hom-Lts) is a ternary Hom-algebra  $(A, [, ], \alpha = (\alpha_1, \alpha_2))$  such that

$$(2.6) \quad [x, y, z] = -[y, x, z],$$

$$(2.7) \quad \odot_{x,y,z} [x, y, z] = 0,$$

and the ternary Hom-Nambu identity (2.5) holds in  $(A, [, ], \alpha = (\alpha_1, \alpha_2))$ .

One notes that when the twisting maps  $\alpha_1, \alpha_2$  are both equal to the identity map  $id$ , then we recover the usual notion of a Lie triple system [13], [29]. Examples of Hom-Lts could be found in [33].

A particular situation, interesting for our setting, occurs when the twisting maps  $\alpha_i$  are all equal,  $\alpha_1 = \alpha_2 = \alpha$  and  $\alpha([x, y, z]) = [\alpha(x), \alpha(y), \alpha(z)]$  for all  $x, y, z$  in  $A$ . The Hom-Lie triple system  $(A, [, ], \alpha)$  is then said to be *multiplicative* [33]. In case of

multiplicativity, the ternary Hom-Nambu identity (2.5) then reads

$$(2.8) \quad [\alpha(x), \alpha(y), [u, v, w]] = [[x, y, u], \alpha(v), \alpha(w)] + [\alpha(u), [x, y, v], \alpha(w)] \\ + [\alpha(u), \alpha(v), [x, y, w]].$$

In [5] a (multiplicative) *Hom-triple system* is defined as a (multiplicative) ternary Hom-algebra  $(A, [, ], \alpha)$  such that (2.6) and (2.7) are satisfied (thus a multiplicative Hom-Lts is seen as a Hom-triple system in which the identity (2.8) holds; observe that this definition of a Hom-triple system is different from the one formerly given in [33], where a Hom-triple system is just the Hom-algebra  $(A, [, ], \alpha)$ ). With this vision of a Hom-triple system, it is shown ([5]) that every multiplicative non-Hom-associative algebra (i.e. not necessarily Hom-associative algebra) has a natural Hom-triple system structure if define  $[x, y, z] := [[x, y], \alpha(z)] - as(x, y, z) + as(y, x, z)$ . We note here that we get the same result if define another ternary operation on a given Hom-algebra. Specifically, we have the following

**Proposition 2.9.** *Let  $\mathcal{A} = (A, *, \alpha)$  be a multiplicative Hom-algebra. Define on  $\mathcal{A}$  the ternary operation*

$$(2.9) \quad (x, y, z) := as^J(y, z, x)$$

*for all  $x, y, z \in A$ . Then  $(A, (, ,), \alpha)$  is a multiplicative Hom-triple system.*

*Proof.* A proof follows from the straightforward checking of the identities (2.6) and (2.7) for " $(, ,)$ " using the commutativity of the Jordan product " $\circ$ ".  $\square$

Since our results here depend on multiplicativity, in the rest of this paper we assume that all Hom-algebras (binary or ternary) are multiplicative and while dealing with the binary operation " $*$ " and where there is no danger of confusion, we will use juxtaposition in order to reduce the number of braces i.e., e.g.,  $xy * \alpha(z)$  means  $(x * y) * \alpha(z)$ .

Various results and constructions related to Hom-Lts are given in [33]. In particular, it is shown that every Lts  $L$  can be twisted along any self-morphism of  $L$  into a multiplicative Hom-Lts. For our purpose we just mention the following result.

**Proposition 2.10** ([33]). *Let  $(A, *)$  be a Malcev algebra and  $\alpha : A \longrightarrow A$  an algebra morphism. Then  $A_\alpha := (A, [, ],_\alpha, \alpha)$  is a multiplicative Hom-Lts, where  $[x, y, z]_\alpha = \alpha(2xy * z - yz * x - zx * y)$ , for all  $x, y, z$  in  $A$ .*

One observes that the product  $[x, y, z] = 2xy * z - yz * x - zx * y$  is the one defined in [14] providing a Malcev algebra  $(A, *)$  with a Lts structure. A construction describing another view of Proposition 2.10 above, will be given in Section 3 (see Proposition 3.4) via Hom-Malcev algebras. For the time being, we point out the following slight generalization of the result above, producing a sequence of multiplicative Hom-Lts from a given Malcev algebra.

**Proposition 2.11.** *Let  $(A, *)$  be a Malcev algebra and  $\alpha : A \longrightarrow A$  an algebra morphism. Let  $\alpha^0 = id$  and, for any integer  $n \geq 1$ ,  $\alpha^n = \alpha \circ \alpha^{n-1}$ . If define on  $A$  a trilinear operation  $[, , ]_{\alpha^n}$  by*

$$[x, y, z]_{\alpha^n} = \alpha^n(2xy * z - yz * x - zx * y)$$

*for all  $x, y, z$  in  $A$ , then  $(A, [, ],_{\alpha^n}, \alpha^n)$  is a multiplicative Hom-Lts.*

*Proof.* Let  $[x, y, z] = 2xy * z - yz * x - zx * y$  and then  $[x, y, z]_{\alpha^n} = \alpha^n([x, y, z])$ . We shall use the fact that  $(A, [, ],)$  is a Lts ([14]). The identities (2.6) and (2.7) for  $[x, y, z]_{\alpha^n}$  are quite obvious. Next,

$$[\alpha^n(x), \alpha^n(y), [u, v, w]_{\alpha^n}]_{\alpha^n} = [\alpha^n(x), \alpha^n(y), \alpha^n([u, v, w])]_{\alpha^n} \\ = \alpha^{2n}([x, y, [u, v, w]])$$

$$\begin{aligned}
&= \alpha^{2n}([x, y, u], v, w) + \alpha^{2n}(u, [x, y, v], w) + \alpha^{2n}(u, v, [x, y, w]) \\
&= [\alpha^n([x, y, u]), \alpha^n(v), \alpha^n(w)]_{\alpha^n} + [\alpha^n(u), \alpha^n([x, y, v]), \alpha^n(w)]_{\alpha^n} \\
&\quad + [\alpha^n(u), \alpha^n(v), \alpha^n([x, y, w])]_{\alpha^n} \\
&= [[x, y, u]_{\alpha^n}, \alpha^n(v), \alpha^n(w)]_{\alpha^n} + [\alpha^n(u), [x, y, v]_{\alpha^n}, \alpha^n(w)]_{\alpha^n} \\
&\quad + [\alpha^n(u), \alpha^n(v), [x, y, w]_{\alpha^n}]_{\alpha^n}
\end{aligned}$$

and so (2.8) holds for  $[\cdot, \cdot]_{\alpha^n}$ . Thus  $(A, [\cdot, \cdot]_{\alpha^n}, \alpha^n)$  is a multiplicative Hom-Lts.  $\square$

In [2] we defined a Hom-Bol algebra as a twisted generalization of a (left) Bol algebra. For the introduction and original studies of Bol algebras, we refer to [21], [26], [27]. Bol algebras are further considered in, e.g., [8], [24].

**Definition 2.12** ([2]). A *Hom-Bol algebra* is a quadruple  $(A, [\cdot, \cdot], (\cdot, \cdot), \alpha)$  in which  $A$  is a vector space,  $[\cdot, \cdot]$  a binary operation,  $(\cdot, \cdot)$  a ternary operation on  $A$ , and  $\alpha : A \rightarrow A$  a linear map such that

$$\begin{aligned}
(\text{HB1}) \quad &\alpha([x, y]) = [\alpha(x), \alpha(y)], \\
(\text{HB2}) \quad &\alpha((x, y, z)) = (\alpha(x), \alpha(y), \alpha(z)), \\
(\text{HB3}) \quad &[x, y] = -[y, x], \\
(\text{HB4}) \quad &(x, y, z) = -(y, x, z), \\
(\text{HB5}) \quad &\circlearrowleft_{x, y, z}(x, y, z) = 0, \\
(\text{HB6}) \quad &(\alpha(x), \alpha(y), [u, v]) = [(x, y, u), \alpha^2(v)] + [\alpha^2(u), (x, y, v)] \\
&\quad + (\alpha(u), \alpha(v), [x, y]) - [[\alpha(u), \alpha(v)], [\alpha(x), \alpha(y)]], \\
(\text{HB7}) \quad &(\alpha^2(x), \alpha^2(y), (u, v, w)) = ((x, y, u), \alpha^2(v), \alpha^2(w)) + \alpha^2(u), (x, y, v), \alpha^2(w) \\
&\quad + (\alpha^2(u), \alpha^2(v), (x, y, w))
\end{aligned}$$

for all  $u, v, w, x, y, z \in A$ .

The identities (HB1) and (HB2) mean the multiplicativity of  $(A, [\cdot, \cdot], (\cdot, \cdot), \alpha)$ . It is built into our definition for convenience.

One observes that for  $\alpha = id$  the identities (HB3)-(HB7) reduce to the defining identities of a (left) Bol algebra [21] (see also [8], [24]). If  $[x, y] = 0$  for all  $x, y \in A$ , then  $(A, [\cdot, \cdot], (\cdot, \cdot), \alpha)$  becomes a (multiplicative) Hom-Lts  $(A, (\cdot, \cdot), \alpha^2)$ .

Construction results and some examples of Hom-Bol algebras are given in [2]. In particular, Hom-Bol algebras can be constructed from Malcev algebras. The Hom-analogues of the construction of Bol algebras from Malcev algebras [21] or from right alternative algebras [22] (see also [8]) are considered in this paper.

### 3. Hom-Lts and Hom-Bol algebras from Hom-Malcev algebras

In this section, we prove that every multiplicative Hom-Malcev algebra has a natural multiplicative Hom-Lts structure (Theorem 3.2) and, moreover, a natural Hom-Bol algebra structure (Theorem 3.5). Theorem 3.2 could be seen as the Hom-analogue of the Loos' result ([14], Satz 1). Besides the identities (2.3) and (2.4), Lemma 3.1 below is a tool in the proof of this result. Theorem 3.5 could be seen as the Hom-analogue of a construction by Mikheev [21] of Bol algebras from Malcev algebras. Proposition 3.4 is another view of a result in [33] (see Proposition 2.10 above).

In his work [14], Loos considered in a Malcev algebra  $(A, *)$ , the following ternary operation:

$$(3.1) \quad \{x, y, z\} = 2xy * z - yz * x - zx * y.$$

Then  $(A, \{, \cdot, \cdot\})$  turns out to be a Lts. This result, in the Hom-algebra setting, looks as in Theorem 3.2 below. Similarly as in the Loos construction, our investigations are based on the following ternary operation in a Hom-Malcev algebra  $(A, *, \alpha)$ :

$$(3.2) \quad \{x, y, z\}_\alpha = 2xy * \alpha(z) - yz * \alpha(x) - zx * \alpha(y).$$

From (3.2) it clearly follows that  $\{, , \}_\alpha$  can also be written as

$$(3.3) \quad \{x, y, z\}_\alpha = -J_\alpha(x, y, z) + 3xy * \alpha(z).$$

One observes that when  $\alpha = id$ , we recover the product (3.1). This agrees with the reduction of the Hom-Malcev algebra  $(A, *, \alpha)$  to the Malcev algebra  $(A, *)$ . First, we prove the following

**Lemma 3.1.** *Let  $(A, *, \alpha)$  be a Hom-Malcev algebra. If define on  $(A, *, \alpha)$  a ternary operation " $\{, , \}_\alpha$ " by (3.2), then*

$$(3.4) \quad \{\alpha(x), \alpha(y), u * v\}_\alpha = \alpha^2(u) * \{x, y, v\}_\alpha + \{x, y, u\}_\alpha * \alpha^2(v) - J_\alpha(\alpha(u), \alpha(v), x * y)$$

for all  $u, v, x, y$  in  $A$ .

*Proof.* Let us write (2.4) as

$$-J_\alpha(\alpha(x), \alpha(y), u * v) = -J_\alpha(x, y, u) * \alpha^2(v) + \alpha^2(u) * (-J_\alpha(x, y, v)) + 3J_\alpha(\alpha(u), \alpha(v), x * y) - J_\alpha(\alpha(u), \alpha(v), x * y)$$

i.e.

$$-J_\alpha(\alpha(x), \alpha(y), u * v) = -J_\alpha(x, y, u) * \alpha^2(v) + \alpha^2(u) * (-J_\alpha(x, y, v)) + 3\alpha(u)\alpha(v) * \alpha(x * y) + 3(\alpha(v) * xy) * \alpha^2(u) + 3(xy * \alpha(u)) * \alpha^2(v) - J_\alpha(\alpha(u), \alpha(v), x * y).$$

Therefore, by multiplicativity, we have

$$\begin{aligned} & -J_\alpha(\alpha(x), \alpha(y), u * v) + 3\alpha(x)\alpha(y) * \alpha(u * v) \\ &= (-J_\alpha(x, y, u) + 3xy * \alpha(u)) * \alpha^2(v) + \alpha^2(u) * (-J_\alpha(x, y, v) + 3xy * \alpha(v)) \\ & \quad - J_\alpha(\alpha(u), \alpha(v), x * y) \end{aligned}$$

and so, we get (3.4) by (3.3). □

We now prove

**Theorem 3.2.** *Let  $(A, *, \alpha)$  be a multiplicative Hom-Malcev algebra. If define on  $(A, *, \alpha)$  a ternary operation " $\{, , \}_\alpha$ " by (3.2), then  $(A, \{, , \}_\alpha, \alpha^2)$  is a multiplicative Hom-Lts.*

*Proof.* We must prove the validity of (2.6), (2.7), (2.8) for the operation (3.2) in the Hom-Malcev algebra  $(A, *, \alpha)$ .

First observe that the multiplicativity of  $(A, *, \alpha)$  implies that  $\alpha^2(\{x, y, z\}_\alpha) = \{\alpha^2(x), \alpha^2(y), \alpha^2(z)\}_\alpha$ , with  $x, y, z$  in  $A$ .

From the skew-symmetry of " $*$ " and  $J_\alpha(x, y, z)$ , it clearly follows from (3.3) that  $\{x, y, z\}_\alpha = -\{y, x, z\}_\alpha$  which is (2.6) for " $\{, , \}_\alpha$ ".

Next, using (3.3) and the skew-symmetry of  $J_\alpha(x, y, z)$  where applicable, we compute

$$\begin{aligned} & \{x, y, z\}_\alpha + \{y, z, x\}_\alpha + \{z, x, y\}_\alpha \\ &= -J_\alpha(x, y, z) + 3xy * \alpha(z) - J_\alpha(y, z, x) + 3yz * \alpha(x) - J_\alpha(z, x, y) + 3zx * \alpha(y) \\ &= -3J_\alpha(x, y, z) + 3J_\alpha(x, y, z) = 0 \end{aligned}$$

and thus  $\bigcirc_{x,y,z} \{x, y, z\}_\alpha = 0$ , so we get (2.7) for " $\{, , \}_\alpha$ ".

Consider now  $\{\alpha^2(x), \alpha^2(y), \{u, v, w\}_\alpha\}_\alpha$  in  $(A, *, \alpha)$ . Then

$$\begin{aligned} & \{\alpha^2(x), \alpha^2(y), \{u, v, w\}_\alpha\}_\alpha \\ &= \{\alpha^2(x), \alpha^2(y), 2uv * \alpha(w) - vw * \alpha(u) - wu * \alpha(v)\}_\alpha \text{ (by (3.2))} \\ &= \{\alpha^2(x), \alpha^2(y), 2uv * \alpha(w)\}_\alpha - \{\alpha^2(x), \alpha^2(y), vw * \alpha(u)\}_\alpha \\ & \quad - \{\alpha^2(x), \alpha^2(y), wu * \alpha(v)\}_\alpha \\ &= \{\alpha(x), \alpha(y), 2u * v\}_\alpha * \alpha^3(w) + \alpha^2(2u * v) * \{\alpha(x), \alpha(y), \alpha(w)\}_\alpha \\ & \quad - J_\alpha(\alpha(2u * v), \alpha^2(w), \alpha(x * y)) - \{\alpha(x), \alpha(y), v * w\}_\alpha * \alpha^3(u) \\ & \quad - \alpha^2(v * w) * \{\alpha(x), \alpha(y), \alpha(u)\}_\alpha + J_\alpha(\alpha(v * w), \alpha^2(u), \alpha(x * y)) \end{aligned}$$

$$\begin{aligned}
& -\{\alpha(x), \alpha(y), w * u\}_\alpha * \alpha^3(v) - \alpha^2(w * u) * \{\alpha(x), \alpha(y), \alpha(v)\}_\alpha \\
& + J_\alpha(\alpha(w * u), \alpha^2(v), \alpha(x * y)) \text{ (by (3.4))} \\
= & (2\{x, y, u\}_\alpha * \alpha^2(v) + 2\alpha^2(u) * \{x, y, v\}_\alpha \\
& - 2J_\alpha(\alpha(u), \alpha(v), x * y)) * \alpha^3(w) + 2\alpha^2(u * v) * \{\alpha(x), \alpha(y), \alpha(w)\}_\alpha \\
& - J_\alpha(\alpha(2u * v), \alpha^2(w), \alpha(x * y)) - (\{x, y, v\}_\alpha * \alpha^2(w) \\
& + \alpha^2(v) * \{x, y, w\}_\alpha - J_\alpha(\alpha(v), \alpha(w), x * y)) * \alpha^3(u) \\
& - \alpha^2(v * w) * \{\alpha(x), \alpha(y), \alpha(u)\}_\alpha + J_\alpha(\alpha(v * w), \alpha^2(u), \alpha(x * y)) \\
& - (\{x, y, w\}_\alpha * \alpha^2(u) + \alpha^2(w) * \{x, y, u\}_\alpha \\
& - J_\alpha(\alpha(w), \alpha(u), x * y)) * \alpha^3(v) - \alpha^2(w * u) * \{\alpha(x), \alpha(y), \alpha(v)\}_\alpha \\
& + J_\alpha(\alpha(w * u), \alpha^2(v), \alpha(x * y)) \text{ (again by (3.4))} \\
= & 2\{x, y, u\}_\alpha \alpha^2(v) * \alpha^3(w) + 2\alpha^2(u) \{x, y, v\}_\alpha * \alpha^3(w) \\
& - 2J_\alpha(\alpha(u), \alpha(v), x * y) * \alpha^3(w) + 2\alpha^2(u * v) * \{\alpha(x), \alpha(y), \alpha(w)\}_\alpha \\
& - J_\alpha(\alpha(2u * v), \alpha^2(w), \alpha(x * y)) - \{x, y, v\}_\alpha \alpha^2(w) * \alpha^3(u) \\
& - \alpha^2(v) \{x, y, w\}_\alpha * \alpha^3(u) + J_\alpha(\alpha(v), \alpha(w), x * y) * \alpha^3(u) \\
& - \alpha^2(v * w) * \{\alpha(x), \alpha(y), \alpha(u)\}_\alpha + J_\alpha(\alpha(v * w), \alpha^2(u), \alpha(x * y)) \\
& - \{x, y, w\}_\alpha \alpha^2(u) * \alpha^3(v) - \alpha^2(w) \{x, y, u\}_\alpha * \alpha^3(v) \\
& + J_\alpha(\alpha(w), \alpha(u), x * y) * \alpha^3(v) - \alpha^2(w * u) * \{\alpha(x), \alpha(y), \alpha(v)\}_\alpha \\
& + J_\alpha(\alpha(w * u), \alpha^2(v), \alpha(x * y)) \\
= & 2\{x, y, u\}_\alpha \alpha^2(v) * \alpha^3(w) - \alpha^2(v * w) * \alpha(\{x, y, u\}_\alpha \\
& - \alpha^2(w) \{x, y, u\}_\alpha * \alpha^3(v) + 2\alpha^2(u) \{x, y, v\}_\alpha * \alpha^3(w) \\
& - \{x, y, v\}_\alpha \alpha^2(w) * \alpha^3(u) - \alpha^2(w * u) * \alpha(\{x, y, v\}_\alpha) \\
& + 2\alpha^2(u * v) * \alpha(\{x, y, w\}_\alpha) - \alpha^2(v) \{x, y, w\}_\alpha * \alpha^3(u) \\
& - \{x, y, w\}_\alpha \alpha^2(u) * \alpha^3(v) - 2J_\alpha(\alpha(u), \alpha(v), x * y) * \alpha^3(w) \\
& - J_\alpha(\alpha(2u * v), \alpha^2(w), \alpha(x * y)) + J_\alpha(\alpha(v), \alpha(w), x * y) * \alpha^3(u) \\
& + J_\alpha(\alpha(v * w), \alpha^2(u), \alpha(x * y)) + J_\alpha(\alpha(w), \alpha(u), x * y) * \alpha^3(v) \\
& + J_\alpha(\alpha(w * u), \alpha^2(v), \alpha(x * y)) \\
& \text{(rearranging terms)} \\
= & \{\{x, y, u\}_\alpha, \alpha^2(v), \alpha^2(w)\}_\alpha + \{\alpha^2(u), \{x, y, v\}_\alpha, \alpha^2(w)\}_\alpha \\
& + \{\alpha^2(u), \alpha^2(v), \{x, y, w\}_\alpha\}_\alpha \\
& + [ - 2(J_\alpha(\alpha(u), \alpha(v), x * y) * \alpha^3(w) + J_\alpha(\alpha(u * v), \alpha^2(w), \alpha(x * y))) \\
& + J_\alpha(\alpha(v), \alpha(w), x * y) * \alpha^3(u) + J_\alpha(\alpha(v * w), \alpha^2(u), \alpha(x * y)) \\
& + J_\alpha(\alpha(w), \alpha(u), x * y) * \alpha^3(v) + J_\alpha(\alpha(w * u), \alpha^2(v), \alpha(x * y))] .
\end{aligned}$$

In this latest expression, denote by  $N(u, v, w, x, y)$  the expression in "[...]" . To conclude, we proceed to show that  $N(u, v, w, x, y) = 0$ .

Observe first that, by (2.3), we have

$$\begin{aligned}
& J_\alpha(\alpha(u), x * y, \alpha(w)) * \alpha^2(\alpha(v)) + J_\alpha(\alpha(v), x * y, \alpha(w)) * \alpha^2(\alpha(u)) \\
= & J_\alpha(\alpha^2(u), \alpha(x * y), \alpha(v) * \alpha(w)) + J_\alpha(\alpha^2(v), \alpha(x * y), \alpha(u) * \alpha(w))
\end{aligned}$$

i.e.,

$$\begin{aligned}
& J_\alpha(\alpha(w), \alpha(u), x * y) * \alpha^3(v) + J_\alpha(\alpha(w * u), \alpha^2(v), \alpha(x * y)) \\
= & J_\alpha(\alpha(v * w), \alpha^2(u), \alpha(x * y)) + J_\alpha(\alpha(v), \alpha(w), x * y) * \alpha^3(u)
\end{aligned}$$

With this observation, the expression  $N(u, v, w, x, y)$  is transformed as follows:

$$N(u, v, w, x, y)$$



$$\begin{aligned}
&= 2[-J_\alpha(\alpha(u), \alpha(v), x * y) * \alpha^3(w) - J_\alpha(\alpha(u * v), \alpha^2(w), \alpha(x * y))] \\
&\quad + 2[J_\alpha(\alpha(v * w), \alpha^2(u), \alpha(x * y)) + J_\alpha(\alpha(v), \alpha(w), x * y) * \alpha^3(u)] \\
&= 2[-J_\alpha(\alpha^2(w), \alpha(x * y), \alpha(u) * \alpha(v)) - J_\alpha(\alpha^2(u), \alpha(x * y), \alpha(w) * \alpha(v)) \\
&\quad - J_\alpha(\alpha(u), \alpha(v), x * y) * \alpha^3(w) + J_\alpha(\alpha(v), \alpha(w), x * y) * \alpha^3(u)] \\
&= 2[-J_\alpha(\alpha(w), x * y, \alpha(v)) * \alpha^3(u) - J_\alpha(\alpha(u), x * y, \alpha(v)) * \alpha^3(w) \\
&\quad - J_\alpha(\alpha(u), \alpha(v), x * y) * \alpha^3(w) + J_\alpha(\alpha(v), \alpha(w), x * y) * \alpha^3(u)] \\
&\quad (\text{applying (2.3) to } -J_\alpha(\alpha^2(w), \alpha(x * y), \alpha(u) * \alpha(v)) \\
&\quad - J_\alpha(\alpha^2(u), \alpha(x * y), \alpha(w) * \alpha(v))) \\
&= 0 \text{ ( by the skew-symmetry of } J_\alpha(x, y, z)).
\end{aligned}$$

Therefore, we obtain that (2.8) holds for " $\{, , \}_\alpha$ " and thus  $(A, \{, , \}_\alpha, \alpha^2)$  is a Hom-Lts.  $\square$

*Remark 3.3.* In the proof of his result, Loos ([14], Satz 1) used essentially the fact that the left translations  $L(x)$  in a Malcev algebra  $(A, *)$  are derivations of the ternary operation " $\{, , \}$ " defined by (3.1). Unfortunately, for Hom-Malcev algebras such a tool is still not available at hand.

From [32] (Theorem 2.12) we know that any Malcev algebra  $A$  can be twisted into a Hom-Malcev algebra along any linear self-map of  $A$ . Consistent with this result, we recall the following method for constructing Hom-Lts which, in fact, is a result in [33] (see also Proposition 2.10 and Proposition 2.11 above) but using a Hom-Malcev algebra construction in our proof (as a consequence of Theorem 3.2).

**Proposition 3.4.** *Let  $(A, *)$  be a Malcev algebra and  $\alpha$  any self-morphism of  $(A, *)$ . If define on  $(A, *)$  a ternary operation " $\{, , \}_\alpha$ " by*

$$\{x, y, z\}_\alpha = \alpha^2(2xy * z - yz * x - zx * y),$$

*then  $(A, \{, , \}_\alpha, \alpha^2)$  is a multiplicative Hom-Lts.*

*Proof.* One knows ([32], Theorem 2.12) that from  $(A, *)$  and any self-morphism  $\alpha$  of  $(A, *)$ , we get a (multiplicative) Hom-Malcev algebra  $(A, \tilde{*}, \alpha)$ , where  $x\tilde{*}y = \alpha(x * y)$  for all  $x, y$  in  $A$ . Next, if define on  $(A, \tilde{*}, \alpha)$  a ternary operation

$$\{x, y, z\}_\alpha := 2(x\tilde{*}y)\tilde{*}\alpha(z) - (y\tilde{*}z)\tilde{*}\alpha(x) - (z\tilde{*}x)\tilde{*}\alpha(y),$$

then by Theorem 3.2,  $(A, \{, , \}_\alpha, \alpha^2)$  is a Hom-Lts and " $\{, , \}_\alpha$ " expresses through " $*$ " as

$$\begin{aligned}
\{x, y, z\}_\alpha &= 2\alpha(\alpha(x * y) * \alpha(z)) - \alpha(\alpha(y * z) * \alpha(x)) - \alpha(\alpha(z * x) * \alpha(y)) \\
&= 2\alpha^2(xy * z) - \alpha^2(yz * x) - \alpha^2(zx * y) \\
&= \alpha^2(2xy * z - yz * x - zx * y).
\end{aligned}$$

$\square$

Observe that, although constructed in a quite different way, the operation " $\{, , \}_\alpha$ " in Proposition 3.4 above coincide with " $[, , ]_{\alpha^n}$ " in Proposition 2.11 for  $n = 2$ .

Combining Lemma 3.1 and Theorem 3.2, we get the following result.

**Theorem 3.5.** *Let  $(A, *, \alpha)$  be a multiplicative Hom-Malcev algebra. If define on  $(A, *, \alpha)$  a ternary operation  $(, , )_\alpha$  by*

$$(3.5) \quad (x, y, z)_\alpha := \frac{1}{3}\{x, y, z\}_\alpha,$$

*where " $\{, , \}_\alpha$ " is defined by (3.3), then  $(A, *, (, , )_\alpha, \alpha)$  is a Hom-Bol algebra.*

*Proof.* The definition (3.5) and Theorem 3.2 imply that  $(A, (, , )_\alpha, \alpha^2)$  is a multiplicative Hom-Lts i.e. (HB2), (HB4), (HB5) and (HB7) hold for  $(A, *, (, , )_\alpha, \alpha)$ . Now, (HB1) and (HB3) are respectively the multiplicativity and skew-symmetry of " $*$ ". Next, we are done if we prove (HB6) for  $(A, *, (, , )_\alpha, \alpha)$ .

From (3.3) and multiplicativity we have  
 $-J_\alpha(\alpha(u), \alpha(v), x * y) = \{\alpha(u), \alpha(v), x * y\}_\alpha - 3(\alpha(u)\alpha(v)) * (\alpha(x)\alpha(y))$   
and then (3.4) takes the form  
 $\{\alpha(x), \alpha(y), u * v\}_\alpha = \{x, y, u\}_\alpha * \alpha^2(v) + \alpha^2(u) * \{x, y, v\}_\alpha$   
 $+ \{\alpha(u), \alpha(v), x * y\}_\alpha - 3(\alpha(u)\alpha(v)) * (\alpha(x)\alpha(y)).$

Multiplying by  $\frac{1}{3}$  each member of this latter equality and using (3.5), we get  
 $(\alpha(x), \alpha(y), u * v)_\alpha = (x, y, u)_\alpha * \alpha^2(v) + \alpha^2(u) * (x, y, v)_\alpha$   
 $+ (\alpha(u), \alpha(v), x * y)_\alpha - (\alpha(u)\alpha(v)) * (\alpha(x)\alpha(y))$   
which is (HB6) for  $(A, *, (, , )_\alpha, \alpha)$ . So  $(A, *, (, , )_\alpha, \alpha)$  is a Hom-Bol algebra.  $\square$

Since any Hom-alternative algebra is Hom-Malcev admissible ([32], Theorem 3.8), from Theorem 3.5 we have the following

**Corollary 3.6.** *Let  $(A, *, \alpha)$  be a multiplicative Hom-alternative algebra. Then  $(A, [, ], (, , )_\alpha, \alpha)$  is a Hom-Bol algebra, where  $(x, y, z)_\alpha := -\frac{1}{3}J_\alpha(x, y, z)$   
 $+ xy * \alpha(z)$ , for all  $x, y \in A$ .*  $\square$

The aim of Section 4 is a generalization of Corollary 3.6 to multiplicative right (or left) Hom-alternative algebras.

Various constructions of Hom-Lts are offered in [33] starting from either Hom-associative algebras, Hom-Lie algebras, Hom-Jordan triple systems, ternary totally Hom-associative algebras, Malcev algebras or alternative algebras. In practice, it is easier to construct Hom-Lts or Hom-Bol algebras from well-known (binary) algebras such as, e.g., alternative algebras or Malcev algebras. From this point of view, our construction results (Theorem 3.2, Proposition 3.4 and Theorem 3.5) have rather a theoretical feature (the extension to Hom-algebra setting of the Loos' result [14] and a result by Mikheev [21]) than a practical method for constructing Hom-Lts or Hom-Bol algebras. However, it could be of some interest to get a Hom-Lts or a Hom-Bol algebra from a given Hom-Malcev algebra without resorting to the corresponding Malcev algebra.

#### 4. Hom-Lts and Hom-Bol algebras from right (or left) Hom-alternative algebras

In this section we prove that every multiplicative right (or left) Hom-alternative algebra has a natural Hom-Bol algebra structure (and, subsequently, a natural Hom-Lts structure). This is the Hom-analogue of a result by Mikheev [22] and by Hentzel and Peresi [8].

First we recall some few basic properties of right Hom-alternative algebras that could be found in [17], [35].

The linearized form of the right Hom-alternative identity  $as(x, y, y) = 0$  is given by the following result.

**Lemma 4.1** ([17]). *If  $(A, *, \alpha)$  is a Hom-algebra, then the following statements are equivalent.*

- (i)  $(A, *, \alpha)$  is right Hom-alternative.
- (ii)  $(A, *, \alpha)$  satisfies

$$(4.1) \quad as(x, y, z) = -as(x, z, y)$$

for all  $x, y, z \in A$ .

(iii)  $(A, *, \alpha)$  satisfies

$$(4.2) \quad \alpha(x) * (yz + zy) = xy * \alpha(z) + xz * \alpha(y)$$

for all  $x, y, z \in A$ .

Observe that if  $(A, *, \alpha)$  is a right Hom-alternative algebra, then  $(A, *^{op}, \alpha)$  is a left Hom-alternative algebra, where  $x *^{op} y := y * x$ . So the mirrors of (4.1) and (4.2) hold for  $(A, *^{op}, \alpha)$ :

$$(4.3) \quad as(x, y, z) = -as(y, x, z)$$

and

$$(4.4) \quad ((x *^{op} y) + (y *^{op} x)) *^{op} \alpha(z) = \alpha(x) *^{op} (y *^{op} z) + \alpha(y) *^{op} (x *^{op} z).$$

Now we have the following

**Lemma 4.2.** *In any multiplicative right Hom-alternative algebra  $(A, *, \alpha)$ , the identity*

$$(4.5) \quad as([u, v], \alpha(x), \alpha(y)) = [as(u, x, y), \alpha^2(v)] + [\alpha^2(u), as(v, x, y)] \\ + as(\alpha(v), \alpha(u), [x, y]) - as(\alpha(u), \alpha(v), [x, y])$$

holds for all  $x, y, z \in A$ .

*Proof.* The identity

$$as(uv, \alpha(x), \alpha(y)) = as(u, x, y)\alpha^2(v) + \alpha^2(u)as(v, x, y) - as(\alpha(u), \alpha(v), [x, y])$$

holds in any right Hom-alternative algebra (see [35], Theorem 7.1 (7.1.1c)). Next, in this identity, switching  $u$  and  $v$ , we have

$$as(vu, \alpha(x), \alpha(y)) = as(v, x, y)\alpha^2(u) + \alpha^2(v)as(u, x, y) - as(\alpha(v), \alpha(u), [x, y]).$$

Then, subtracting memberwise this latter equality from the one above and using the linearity of  $as$ , we get (4.5).  $\square$

Note that in case when  $(A, *, \alpha)$  is a left Hom-alternative algebra, the identity (4.5) reads as

$$(4.6) \quad as(\alpha(x), \alpha(y), [u, v]) = [as(x, y, u), \alpha^2(v)] + [\alpha^2(u), as(x, y, v)] \\ + as([x, y], \alpha(v), \alpha(u)) - as([x, y], \alpha(u), \alpha(v)).$$

In any multiplicative right (or left) Hom-alternative algebra  $(A, *, \alpha)$  we consider the ternary operation defined by (2.9), i.e.

$$(x, y, z) := as^J(y, z, x),$$

where  $as^J$  is the Hom-Jordan associator defined in Section 2. Observe that for  $\alpha = id$  the ternary operation " $(, , )$ " is precisely the one defined in [8] (see also [22], Remark 2) and that makes any right (or left) alternative algebra into a left Bol algebra. In [8], Hentzel and Peresi used the approach of Mikheev [22] who formerly proved that the commutator algebra of any right alternative algebra has a left Bol algebra structure.

**Proposition 4.3.** *(i) If  $(A, *, \alpha)$  is a multiplicative right Hom-alternative algebra, then*

$$(4.7) \quad (x, y, z) = [[x, y], \alpha(z)] - 2as(z, x, y)$$

for all  $x, y, z \in A$ .

(ii) If  $(A, *, \alpha)$  is a multiplicative left Hom-alternative algebra, then

$$(4.8) \quad (x, y, z) = [[x, y], \alpha(z)] - 2as(x, y, z)$$

for all  $x, y, z \in A$ .

*Proof.* (i) From (2.9) we have

$$\begin{aligned} (x, y, z) &= (y \circ z) \circ \alpha(x) - \alpha(y) \circ (z \circ x) \\ &= ((y * z) + (z * y)) * \alpha(x) + [\alpha(x) * ((y * z) + (z * y))] \\ &\quad - [\alpha(y) * ((z * x) + (x * z))] - ((z * x) + (x * z)) * \alpha(y) \\ &= ((y * z) + (z * y)) * \alpha(x) + [(x * y) * \alpha(z) + (x * z) * \alpha(y)] \\ &\quad - [(y * z) * \alpha(x) + (y * x) * \alpha(z)] \\ &\quad - ((z * x) + (x * z)) * \alpha(y) \quad (\text{by (4.2)}) \\ &= (z * y) * \alpha(x) + (x * y) * \alpha(z) - (y * x) * \alpha(z) - (z * x) * \alpha(y) \\ &= (z * y) * \alpha(x) - (z * x) * \alpha(y) + [x, y] * \alpha(z) \\ &= (z * y) * \alpha(x) - (z * x) * \alpha(y) + [[x, y], \alpha(z)] + \alpha(z) * [x, y] \\ &= [[x, y], \alpha(z)] + (z * y) * \alpha(x) - \alpha(z) * (y * x) - (z * x) * \alpha(y) \\ &\quad + \alpha(z) * (x * y) \\ &= [[x, y], \alpha(z)] + as(z, y, x) - as(z, x, y) \\ &= [[x, y], \alpha(z)] - 2as(z, x, y) \quad (\text{by (4.1)}) \end{aligned}$$

and so we get (4.7).

(ii) Proceeding as above but using (4.4) and then (4.3), one gets (4.8).  $\square$

We are now in position to prove the main result of this section.

**Theorem 4.4.** *Let  $(A, *, \alpha)$  be a multiplicative right (resp. left) Hom-alternative algebra. If define on  $A$  a ternary operation " $(, , )$ " by (4.7) (resp. (4.8)), then  $(A, (, , ), \alpha^2)$  is a Hom-Lts and  $(A, [, ], (, , ), \alpha)$  is a Hom-Bol algebra.*

*Proof.* We prove the theorem for a multiplicative right Hom-alternative algebra  $(A, *, \alpha)$  (the proof of the left case is the mirror of the right one).

The identities (HB1) and (HB2) follow from the multiplicativity of  $(A, *, \alpha)$ . The identities (HB3) and (HB4) are obvious from the definition of " $[, ]$ " and " $(, , )$ ". The identity (HB5) follows from Proposition 2.9.

In [34] Yau showed that if, on a multiplicative Hom-Jordan algebra  $(A, \circ, \alpha)$ , define a ternary operation by

$$[x, y, z] := 2(\alpha(x) \circ (y \circ z) - \alpha(y) \circ (x \circ z)),$$

then  $(A, [, ], \alpha^2)$  is a multiplicative Hom-Lts (see [34], Corollary 4.1). Now, observe that  $[x, y, z] = 2as^J(y, z, x)$ , i.e.  $[x, y, z] = 2(x, y, z)$ . Therefore, since every multiplicative right Hom-alternative algebra is Hom-Jordan admissible (see [35], Theorem 4.3), we conclude that  $(A, (, , ), \alpha^2)$  is a multiplicative Hom-Lts and so the identity (HB7) holds for  $(A, [, ], (, , ), \alpha)$ .

Next,  $(A, [, ], (, , ), \alpha)$  is a Hom-Bol algebra if we prove that (HB6) additionally holds.

Write (4.7) as

$$(4.9) \quad -2as(z, x, y) = (x, y, z) - [[x, y], \alpha(z)].$$

Multiplying each member of (4.5) by  $-2$  and next using (4.9), we get

$$\begin{aligned} &(\alpha(x), \alpha(y), [u, v]) - [[\alpha(x), \alpha(y)], \alpha([u, v])] \\ &= [(x, y, u) - [[x, y], \alpha(u)], \alpha^2(v)] + [\alpha^2(u), (x, y, v) - [[x, y], \alpha(v)]] \end{aligned}$$

$$\begin{aligned}
& +(\alpha(u), [x, y], \alpha(v)) - [[\alpha(u), [x, y]], \alpha^2(v)] \\
& -(\alpha(v), [x, y], \alpha(u)) + [[\alpha(v), [x, y]], \alpha^2(u)] \\
& \text{i.e.}
\end{aligned}$$

$$\begin{aligned}
(4.10) \quad & (\alpha(x), \alpha(y), [u, v]) = [(x, y, u), \alpha^2(v)] + [\alpha^2(u), (x, y, v)] \\
& -([x, y], \alpha(u), \alpha(v)) + ([x, y], \alpha(v), \alpha(u)) + \alpha([[x, y], [u, v]]).
\end{aligned}$$

Observe that  $-([x, y], \alpha(u), \alpha(v)) + ([x, y], \alpha(v), \alpha(u))$   
 $= (\alpha(u), [x, y], \alpha(v)) + ([x, y], \alpha(v), \alpha(u))$   
 $= -(\alpha(v), \alpha(u), [x, y])$  (since  $\odot_{a,b,c}(a, b, c) = 0$  by (HB5))  
 $= (\alpha(u), \alpha(v), [x, y])$ .

Therefore, (4.10) now reads

$$\begin{aligned}
(\alpha(x), \alpha(y), [u, v]) &= [(x, y, u), \alpha^2(v)] + [\alpha^2(u), (x, y, v)] \\
& + (\alpha(u), \alpha(v), [x, y]) - \alpha([[u, v], [x, y]])
\end{aligned}$$

and so (HB6) holds for  $(A, [, ], (, ), \alpha)$ . Thus we conclude that  $(A, [, ], (, ), \alpha)$  is a Hom-Bol algebra. One gets the same result in case when  $(A, *, \alpha)$  is a multiplicative left Hom-alternative algebra and essentially using (4.8) and (4.6). This finishes the proof.  $\square$

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